Soluble one-dimensional particle conservation models with infinitely many absorbing states

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Abstract
Particle conservation lattice-gas models with infinitely many absorbing states are studied on a one-dimensional lattice. As one increases the particle density, they exhibit a phase transition from an absorbing to an active phase. The models are solved exactly by the use of the transfer matrix technique from which the critical behavior was obtained. We have found that the exponent related to the order parameter, the density of active sites, is 1 for all studied models except one of them with exponent 2.

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1. Introduction

Lattice models with infinitely many absorbing states with particle conservation [1–15] are attractive due to their close connection [2, 16–19] to self-organized criticality (SOC) [20–22]. They are characterized by displaying a continuous phase transition from an absorbing to an active state as one increases the density of particles, which represents a nondiffusive conserved field. At low density the system is trapped into one of the many absorbing configurations. Above a certain finite critical density of particles the system presents an active state in which the density of active sites is nonzero. The critical behavior places these models into a distinct universality class, namely that of absorbing phase transition with a nondiffusive conserved field [1, 3, 9, 10, 12, 13].

The connection to SOC can be understood by adding or removing a particle from the system according to the following procedure [14]: at each sufficiently large time interval, a particle is added if the system has fallen into an absorbing state, a particle is removed if the system is still in the active state. Using this procedure the system will approach by itself the critical state, no matter what the initial number of particles is, a property that characterizes the SOC phenomenon. An avalanche is simply interpreted as the movements of the active particles. Its lifetime can be measured by the time it takes for the system to fall into an

absorbing state. In the subcritical regime, the avalanche lifetime is finite becoming infinite at the critical point and in the supercritical regime.

Among models that display infinitely many absorbing states with particle conservation we find the fixed energy sandpile (FES) models [2, 4, 19], which are variants of the Manna sandpile model [22], the conserved threshold transfer process (CTTP) [1, 2, 10] also known as sandpile models with height restrictions [8], and the conserved lattice-gas (CLG) models [1, 5–7, 14, 15]. The CLG models are exclusion models in which each site of the lattice can be occupied at most by one particle. Depending on the configuration of its neighborhood, a particle may be either active or inactive. Only active particles are allowed to move. If all particles are inactive, the system is trapped into an absorbing state. The original CLG model is the one in which a particle is active if it has at least one nearest neighbor site occupied. Our interest here rests on a class of one-dimensional models that are generalizations of the CLG models.

In one dimension, the original CLG model, which we call the simplest, can be solved exactly [14, 23]. The purpose of this paper is to show that other similar one-dimensional lattice gas models with infinitely many absorbing states, which are generalizations of the simplest CLG model, can be solved exactly by a method to be explained shortly. In addition to the simplest model three others are introduced and solved exactly by the same method. One motivation for considering these other models is to show that the method we use here actually solves a class of models. The models that can be solved by the present approach are those whose stochastic dynamic rules drive the system to a subset of configurations from which it cannot escape with the property that the configurations inside the subset are visited with equal probability.

The exact solution is obtained in three steps. First, we show that in the active state all configurations are equally probable, making up thus a microcanonical ensemble. Second, we change to the equivalent grand canonical ensemble by the introduction of an auxiliary quantity, the activity. Third, the exact solution is obtained by the use of a transfer matrix technique. From the exact solution we find that all models studied here have the same exponent $\beta = 1$ except one of them with $\beta = 2$.

The fact that all configurations of the active state have the same probability of being visited allows us to regard the system as being in equilibrium. However, according to common wisdom [24] there is no phase transition in one dimension, a result that can be proved by applying the Perron–Frobenius theorem to the transfer matrix. Since the models we study here do actually have a phase transition this result seems paradoxical. To solve the paradox it suffices to remember that in the present case all the entries of the transfer matrix are allowed to vanish, and this will occur at a finite density of particles, so that the Perron–Frobenius can no longer be invoked. We remark that in an ordinary one-dimensional equilibrium system, on the other hand, the entries of the transfer matrix are strictly positive, except at zero temperature.

2. The simplest model

Let us consider a one-dimensional lattice with L sites and periodic boundary conditions. Each site of the lattice can either be empty or occupied by just one particle and the stochastic rules are such that the total number N of particles is conserved. Depending on the configuration of its neighborhood a particle may be active or inactive. Active particles move around according to stochastic rules to be specified shortly. Inactive particles do not move but may become active due to the alteration of the configuration of its neighborhood. For a sufficiently small number of particles the stationary state is an absorbing state which is a state with no active particles. In this regime the system becomes trapped in one of the many absorbing states. For

a sufficiently great number of particles however the stationary state is an active state, that is, a state with a nonzero density of active particles.

Let us denote by $\rho = N/L$ the density of particles and by ρ_c the critical density below which the density of active sites ρ_a vanishes. Above the critical density, $\rho > \rho_c$, the rules are such that the system representative point in configuration space evolves in time and eventually enters a subspace Ω_N from which it cannot escape. Once inside this subspace the configurations in Ω_N are visited with equal probability. Since the configurations are equally probable they make up a microcanonical ensemble in Ω_N from which we may determine the stationary properties. However, for the purpose of analysis it is more convenient to change to a grand canonical ensemble which in the thermodynamic limit gives the same properties as the microcanonical ensemble. Using this equivalent grand canonical ensemble the one-dimensional model is solved exactly by the use of the transfer matrix.

We start by examining the simplest soluble model which is defined as follows. A particle is active if one of its two nearest neighbor sites is occupied and the other is empty. Otherwise, it is inactive. At each time step a particle is chosen at random. If it is active it moves to the nearest neighbor empty site. If it is inactive it does not move. Denoting by 0 and 1 an empty and an occupied site, respectively, the possible transitions are represented as follows:

$$110 \rightarrow 101, \qquad 011 \rightarrow 101, \tag{1}$$

and they occur with probability 1. From these two rules it follows immediately that the number N_{11} of pairs of nearest neighbor occupied sites and the number N_{00} of pairs of nearest neighbor empty sites never increase.

We consider first the case in which the number of particles is smaller than $L/2$. In this case it is always possible to construct absorbing configuration by putting particles into the sites of the lattice in such a way that no particle is close to another one so that $N_{11} = 0$ as shown below:

$$10100010100101000100. \tag{2}$$

A configuration of this type will be the end of the system trajectory in configuration space as long as $N < L/2$. To see this it suffices to look at the possible transitions related to four sites in a row which are

$$1100 \rightarrow 1010, \qquad 0011 \rightarrow 0101, \tag{3}$$

$$1101 \rightarrow 1011, \qquad 1011 \rightarrow 1101. \tag{4}$$

They have the same probability and are equivalent to rules in (1). The transitions in (4) conserve both the number of 11 and 00 pairs whereas the transitions in (3) decrease the number of 11 pairs as well as the number of 00 pairs. Taking into account that $N_{00} \neq 0$, because $N < L/2$, configurations of the type 1100 or 0011 will always be present as long as $N_{11} \neq 0$. But the transitions in (3) say that N_{11} should decrease. By this argument the quantity N_{11} will decrease and eventually will reach its minimum value, namely $N_{11} = 0$.

When $N > L/2$ there is no absorbing configuration because in this case there is at least one pair of nearest neighbor occupied sites, and therefore at least two active sites. The stationary state is therefore an active state with $N_{11} \neq 0$. The time evolution is such that N_{11} decreases toward its stationary value which is nonzero. On the other hand N_{00} decreases and eventually reaches the minimum value in the stationary active state which we assume to be $N_{00} = 0$. This means to say that the stationary active state is devoid of 00 pairs of sites. Therefore, the subspace Ω_N of the active state is composed of configurations made up by isolated vacancies as given by the example,

$$11010101101101101011. \tag{5}$$

3

In the subspace Ω_N of active configurations with N particles, we need to consider only the rules in (4) which are the only ones that involve isolated particles. Since the two transitions in (4) are the reverse of each other and they have the same rate, the stochastic process defined on Ω_N obeys detailed balance. It follows then that the stationary probability is the same for all configurations in Ω_N provided the stochastic process is ergodic. This defines a microcanonical ensemble with a fixed number of particles.

We have to show now that the system is ergodic within the subspace Ω_N. This means to prove that any configuration in Ω_N can be reached by rules (4) from any other one. This can be proved by showing that any configuration can reach and can be reached from a certain standard configuration. We choose the standard configuration to be the one constructed by placing zeros at the even sites of the one-dimensional lattice starting from site $i = 2$ until $i = 2(L - N)$ as shown below:

$$1010101010101011111. \tag{6}$$

Using the transitions in (4) we can reach a given configuration, such as that in (5), by shifting in sequence each 'zero' of the standard configuration to its final place starting from the rightmost 'zero'. Since the rules in (4) are the reverse of each other, the trajectory in configuration space can be reversed to reach the standard configuration from the given configuration.

3. A grand canonical ensemble

The properties of the active state of the simplest model defined in the previous section can be determined by the use of a grand canonical ensemble in which the number of particles will be a fluctuating variable. We begin by introducing the grand canonical partition function

$$Z = \sum_n W_n z^n, \tag{7}$$

where z is the activity and W_n denotes the number of configurations in Ω_n. Note that $n \geqslant L/2$ so that we set $W_n = 0$ when $n < L/2$.

Let us denote by Ω the space of configurations of type (5) independently of n. In other words, Ω is the union of all subspaces Ω_n. The probability of a configuration $\mathcal{C}$ belonging to Ω in the grand canonical ensemble is

$$P(\mathcal{C}) = \frac{1}{Z} z^n, \tag{8}$$

where n is the number of particles in configuration $\mathcal{C}$. The partition function Z can be calculated by

$$Z = \sum_{\mathcal{C}} z^n, \tag{9}$$

where the summation is over all configurations $\mathcal{C}$ in Ω.

The partition function is determined by the use of a transfer matrix T connecting two consecutive sites of the one-dimensional lattice. Using periodic boundary conditions the partition function is determined by

$$Z = \text{Tr}\, T^L, \tag{10}$$

where T is a 2×2 matrix. The transfer matrix is set up by taking into account that double vacancy is forbidden, from which it follows that $T(00) = 0$, and by associating a weight $z^{1/2}$ with each occupied site, from which it follows that $T(11) = z$, $T(10) = T(01) = z^{1/2}$.

In the thermodynamic limit

$$\frac{1}{L} \ln Z = \ln \lambda, \tag{11}$$

where λ is the largest eigenvalue of T, solution of

$$\lambda^2 - z\lambda - z = 0, \tag{12}$$

and given by

$$\lambda = \frac{z + \sqrt{z^2 + 4z}}{2}. \tag{13}$$

The average number of particles $\langle n \rangle$ can be calculated by means of (8) and (9). From them it follows that the density of particles $\rho = \langle n \rangle / L$ can be determined from

$$\rho = \frac{z}{\lambda} \frac{\partial \lambda}{\partial z}. \tag{14}$$

This formula gives the following relation between the density of particles ρ and the activity z:

$$\rho = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{z}{z+4}} = \frac{\lambda + 1}{\lambda + 2}. \tag{15}$$

The critical density $\rho_c = 1/2$ is obtained when $z \to 0$.

To determine the density ρ_{11} of the pairs 11 we use the relations $\rho_{11} + \rho_{10} = \rho_1 = \rho$ and $\rho_{10} + \rho_{00} = \rho_0 = 1 - \rho$. Taking into account that $\rho_{00} = 0$ it follows that $\rho_{11} = 2\rho_1 - 1$, that is, $\rho_{11} = 2\rho - 1$.

The transfer matrix approach allows us to determine other quantities such as the density ρ_a of active states given by $\rho_a = \rho_{110} + \rho_{011}$. This quantity can be determined by using a pseudo-matrix (see the appendix) associated with three consecutive sites and defined by $T(000) = T(001) = T(100) = 0$, $T(010) = z^{1/3}$, $T(101) = z^{2/3} T(011) = T(110) = z^{2/3}h$, where h is a parameter, and $T(111) = z$. The density of active sites can then be determined by

$$\rho_a = \frac{h}{\lambda} \frac{\partial \lambda}{\partial h}, \tag{16}$$

calculated at $h = 1$, where λ is the largest eigenvalue of T. Now, an eigenvalue of T is related to h and z by

$$(\lambda^2 - z)(\lambda - z) = z^2 h^2, \tag{17}$$

from which we may obtain the derivative of λ with respect to h. After some straightforward algebraic steps and taking into account the relation (12) we arrive at the following relation between ρ_a and λ:

$$\rho_a = \frac{2\lambda}{(\lambda + 1)(\lambda + 2)}, \tag{18}$$

from which follows the relation between the density of active particles and the density of particles

$$\rho_a = \frac{2}{\rho}(2\rho - 1)(1 - \rho), \tag{19}$$

shown in figure 1. Therefore the order parameter ρ_a vanishes at the critical density $\rho_c = 1/2$ with an exponent $\beta = 1$.

The correlation length ξ can also be determined from the ratio between the two eigenvalues of T. It is given by $\xi^{-1} = |\ln|\lambda'|/\lambda|$ where λ' is the other eigenvalue. As one approaches the critical point, $z \to 0$, we get $\xi = z^{-1/2}$ so that

$$\xi = \frac{1}{4}\left(\rho - \frac{1}{2}\right)^{-1}, \tag{20}$$
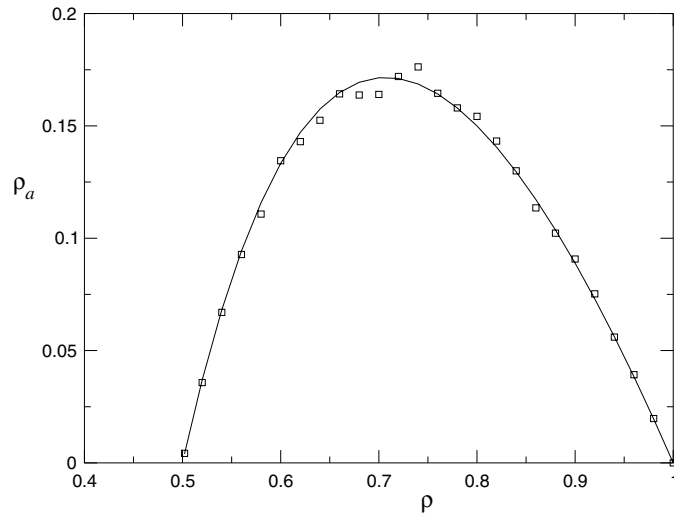
given the exponent $\nu = 1$.

Figure 1. Density ρ_a of active particles as a function of the density ρ of particles for the model 1 as given by equation (19) (continuous line) and by simulation (square symbols).

4. Second model

The possible transitions for the second model are as follows:

$$1100 \rightarrow 1001, \qquad 0011 \rightarrow 1001, \tag{21}$$

$$101 \rightarrow 011, \qquad 101 \rightarrow 110. \tag{22}$$

The two transitions in (21) occur with probability 1 each and describe a jump of a particle to the next-nearest-neighbor site. Each transition in (22) occurs with a probability 1/2. An absorbing configuration is any configuration without isolated vacancies and such that the particles are isolated as shown below:

$$10010001001000010000. \tag{23}$$

Since any cluster of 'zeros' has two or more 'zeros', such an absorbing configuration can be constructed provided $N < L/3$.

When $N > L/3$, either the system displays at least one isolated, vacant site, which produces by (22) an 11 pair, or the system already displays an 11 pair so that the stationary state should be active. In this regime, after a transient, the representative point in configuration space will find itself inside the subspace Ω_N of configurations with N particles of the type

$$10011001001110010011, \tag{24}$$

in which the 'zeros' occur only in clusters of two sites. Within the space Ω_N the transitions in (22) are no longer effective and the transitions in (21) reduce to the following ones

$$11001 \rightarrow 10011, \qquad 10011 \rightarrow 11001, \tag{25}$$

which represent diffusion of a 00 pair and are the reverse of each other. The system becomes trapped inside the subspace Ω_N of the active configurations. The ergodicity inside the subspace Ω_N can be shown as before by defining a standard configuration such that the 00 pairs of sites are all to the left as shown below,

$$10010010010010011111. \tag{26}$$

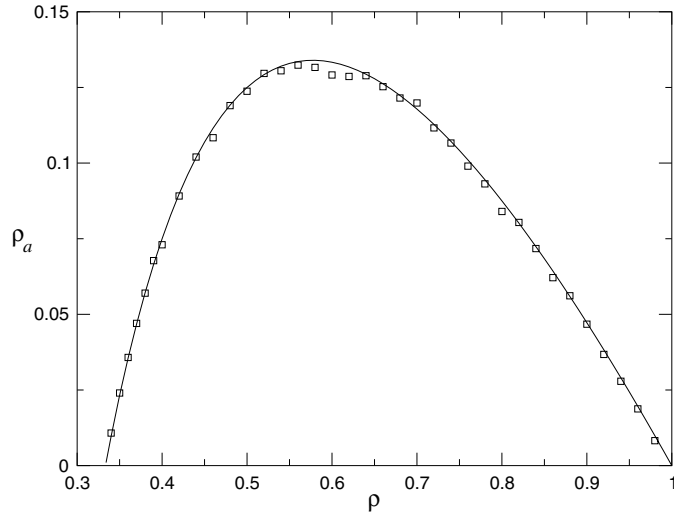**Figure 2.** Density ρ_a of active particles as a function of the density ρ of particles for the model 2 as given by equation (30) (continuous line) and by simulation (square symbols).

Again by diffusion of the 00 pairs, provided by rules in (25), any configuration in Ω_N can reach or can be reached from the standard configuration (26).

We use again the grand canonical ensemble and the transfer matrix approach to determine the properties of the active state. The grand canonical partition function is given by equation (10) and now the pseudo-matrix T involves three consecutive sites. With each occupied site we associate a weight $z^{1/3}$. The elements of T are then: $T(000) = 0$, since a cluster of three 'zeros' in a row is forbidden, $T(001) = T(010) = T(100) = z^{1/3}$, $T(011) = T(110) = z^{2/3}$, $T(101) = 0$ since an isolated vacancy is forbidden, and $T(111) = z$. The eigenvalues of T obey the following relation:

$$z = \frac{\lambda^3}{\lambda^2 + 1}, \tag{27}$$

which used in (14) gives

$$\rho = \frac{\lambda^2 + 1}{\lambda^2 + 3}. \tag{28}$$

The critical density is $\rho_c = 1/3$, obtained by taking the limit $z \to 0$ or $\lambda \to 0$.

The density of active sites ρ_a is given by $\rho_a = \rho_{0011} + \rho_{1100}$. Since there are no isolated vacancies in the active state, $\rho_{1100} = \rho_{110}$ and $\rho_{0011} = \rho_{011}$, so that $\rho_a = \rho_{110} + \rho_{011}$. This last quantity can be determined by modifying the pseudo-matrix T as follows. All elements of T are the same except $T(110)$ and $T(011)$ which now read $T(110) = T(011) = z^{2/3}h$. The density ρ_{110} is then determined by equation (16), calculated at $h = 1$. After straightforward algebraic steps we arrive at the result

$$\rho_{110} = \frac{\lambda^2}{(\lambda^2 + 3)(\lambda^2 + 1)}. \tag{29}$$

Using (28) we finally get

$$\rho_a = 2\rho_{110} = \frac{(3\rho - 1)(1 - \rho)}{2\rho}, \tag{30}$$

plotted in figure 2. Again the order parameter ρ_a vanishes with an exponent $\beta = 1$.

5. Third model

In this model the stochastic rules are as follows:

$$1101 \rightarrow 1011, \qquad 1011 \rightarrow 1101, \tag{31}$$

$$1100 \rightarrow 1001, \qquad 0011 \rightarrow 1001, \tag{32}$$

$$1100 \rightarrow 1010, \qquad 0011 \rightarrow 0101, \tag{33}$$

$$10101 \rightarrow 10011, \qquad 10101 \rightarrow 11001, \tag{34}$$

where the transitions in (32), (33) and (34) occur with probability 1/2 and the two in (31) with probability 1.

When $N < 2L/5$ an absorbing configuration is the one in which the occupied sites are isolated and the clusters of vacancies can be of any size. However, an isolated particle must have at least one next-nearest-neighbor site vacant, so that the sequence 10101 is not allowed, as shown in the example below:

$$100101000100101001000100. \tag{35}$$

The absorbing configuration with the largest density occurs when $N = 2L/5$.

When $N > L/3$ the system may enter a stationary active state characterized by configurations of the type

$$100110101110010110100111, \tag{36}$$

in which the vacant sites are either isolated or appear in pairs. In the active state the transitions in (32), (33) and (34) reduce to the following transitions:

$$11001 \rightarrow 10011, \qquad 10011 \rightarrow 11001, \tag{37}$$

$$11001 \rightarrow 10101, \qquad 10011 \rightarrow 10101, \tag{38}$$

$$10101 \rightarrow 10011, \qquad 10101 \rightarrow 11001. \tag{39}$$

Note that for each given transition the inverse also occurs with the same probability. In the present model the pseudo-matrix T involves three consecutive sites. The elements of T are $T(000) = 0$, since a cluster of three 'zeros' in a row is forbidden, $T(001) = T(010) = T(100) = z^{1/3}$, $T(011) = T(101) = T(110) = z^{2/3}$ and $T(111) = z$. The eigenvalues of T obey the following relation:

$$z = \frac{\lambda^3}{\lambda^2 + \lambda + 1}. \tag{40}$$

Using equation (14) we get the density as a function of λ:

$$\rho = \frac{\lambda^2 + \lambda + 1}{\lambda^2 + 2\lambda + 3}. \tag{41}$$

When $z \rightarrow 0$ or $\lambda \rightarrow 0$ we obtain the critical density, namely $\rho_c = 1/3$.

We remark that for this model the system may also have an absorbing configuration of the type (35) in the interval $1/3 < \rho < 2/5$. However, in this interval it is possible to show that the absorbing state is unstable.

The density of active sites ρ_a is obtained by $\rho_a = 2\rho_{1100} + 2\rho_{1101} + \rho_{10101}$ or $\rho_a = 2\rho_{110} + \rho_{10101}$. The quantity ρ_{110} can be determined by modifying the element $T(110)$ which
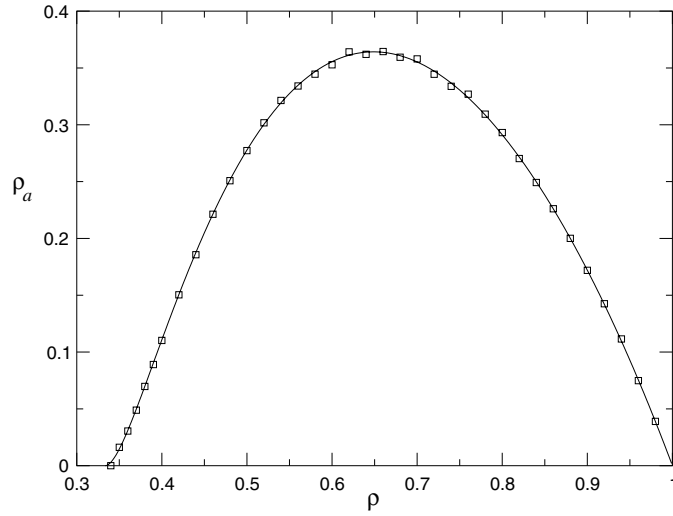
Figure 3. Density ρ_a of active particles as a function of the density ρ of particles for the model 3 as given by equation (44) (continuous line) and by simulation (square symbols).

now reads $T(110) = z^{2/3}h$. The density ρ_{110} is determined by the relation $\rho_{110} = h\partial \ln \lambda / \partial h$ calculated at $h = 1$. After straightforward algebraic steps we arrive at the result

$$\rho_{110} = \frac{\lambda^2(\lambda + 1)}{(\lambda^2 + 2\lambda + 3)(\lambda^2 + \lambda + 1)}. \tag{42}$$

The calculation of ρ_{10101} needs the introduction of a pseudo-matrix involving five sites in a row. By using such a matrix we may find, after straightforward but cumbersome algebraic steps, the result

$$\rho_{10101} = \frac{\lambda^2}{(\lambda^2 + 2\lambda + 3)(\lambda^2 + \lambda + 1)}, \tag{43}$$

so that

$$\rho_a = \frac{\lambda^2(2\lambda + 3)}{(\lambda^2 + 2\lambda + 3)(\lambda^2 + \lambda + 1)}. \tag{44}$$

This equation together with equation (41) gives the density of active sites ρ_a as an implicit function of the density of particles ρ, as shown in figure 3.

Around the critical point $\lambda = 0$, which implies $\rho = \rho_c = 1/3$, the density of particles and the density of active sites behave as $(\rho - \rho_c) = \lambda/9$ and $\rho_a = \lambda^2/3$ so that the density of active sites vanishes as

$$\rho_a = 27(\rho - \rho_c)^2. \tag{45}$$

Therefore, for this model the exponent $\beta = 2$, distinct from that of the previous models.

6. Fourth model

The transitions of the fourth model are

$$1101 \rightarrow 1011, \qquad 1011 \rightarrow 1101, \tag{46}$$

$$11001 \rightarrow 10011, \qquad 10011 \rightarrow 11001, \tag{47}$$

$$11000 \rightarrow 10100, \qquad 00011 \rightarrow 00101, \tag{48}$$

$$11000 \rightarrow 10010, \qquad 00011 \rightarrow 01001, \tag{49}$$

$$10100 \rightarrow 10010, \qquad 00101 \rightarrow 01001. \tag{50}$$

Those in (48) and (49) occur with probability $1/2$ and the others with probability 1. An example of an absorbing configuration is

$$10010001001000010000, \tag{51}$$

in which the particles are isolated and there are no isolated vacancies. Such an absorbing configuration can always be constructed as long as $N < L/3$.

When $N > L/2$ the system must display at least one 11 pair so that by rules in (46), (47), (48) and (49) the system finds itself in an active state. In the interval $L/3 < N < L/2$, either (a) there exists at least one 11 pair and one may apply one of the rules in (46), (47), (48) and (49) or (b) there is at least one isolated vacancy and we may use rules in (50). Therefore, when $N > L/3$ the system is in an active state with the exception of the case $N = L/2$ (if L is even).

Let us consider next a configuration of the type

$$110011010111001110011010, \tag{52}$$

in which the clusters of 'zeros' are either isolated or appear in pairs. The application of rules (46), (47) and (50) will generate configurations of the same type. For configurations of this type the rules in (48) and (49) do not apply and the other transitions reduce to

$$1101 \rightarrow 1011, \qquad 1011 \rightarrow 1101, \tag{53}$$

$$11001 \rightarrow 10011, \qquad 10011 \rightarrow 11001, \tag{54}$$

$$101001 \rightarrow 100101, \qquad 100101 \rightarrow 101001. \tag{55}$$

Note that the transitions appearing in each equation are reverse of each other.

The transition rules in (53), (54) and (55) imply a conservation of the number N_{101} of isolated vacancies and of the number N_{1001} of pair of vacancies. However, these two quantities are not independent because the number N_0 of vacancies is also a conserved quantity and $N_{101} + 2N_{1001} = N_0$. Let us denote by $\Omega_{\ell n}$ the subspace of configurations of the type (52) with ℓ pair of vacancies and n particles. Depending on the initial condition the system will end up in one of the possible subspaces $\Omega_{\ell n}$. Since the transitions within the subspace $\Omega_{\ell n}$ are reverse of each other we may assume that the stationary probability is the same for any configurations in subspace $\Omega_{\ell n}$ provided the system is ergodic within this subspace. But this can be proven by using a reasoning similar to that employed before.

We now change to the grand canonical ensemble defined over the subspace Ω defined as the union of all subspaces $\Omega_{\ell n}$. In the present case we must use another type of activity besides that related to the number of particles. Accordingly, we introduce the activity y related to the number of pairs of vacancies. The grand partition function then reads

$$Z = \sum_{\ell,n} W_{\ell n} y^{\ell} z^{n}, \tag{56}$$

where $W_{\ell n}$ denotes the number of configurations in $\Omega_{\ell n}$.

Again we may use the transfer matrix approach to calculate the partition function. The pseudo-matrix T for the present model involves three consecutive sites and their elements T are $T(000) = 0$, since a cluster of three consecutive 'zeros' is forbidden,
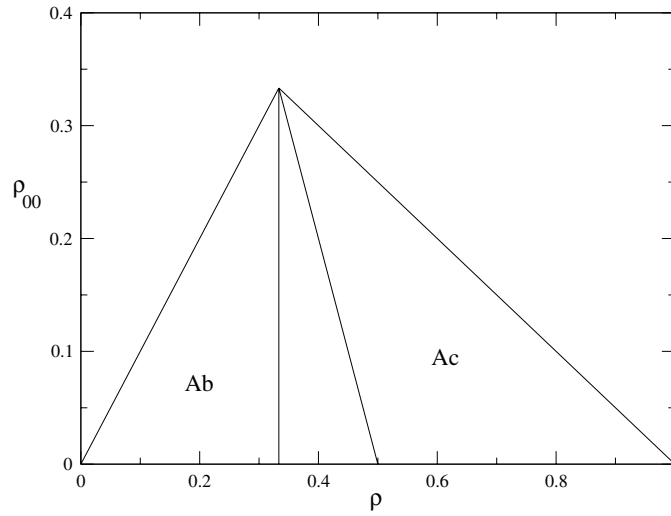
Figure 4. Phase diagram for model 4 in the space ρ_{00} versus ρ. In the stationary state the system is found either in the active (Ac) state or in the absorbing (Ab) state.

$T(001) = T(100) = y^{1/2}z^{1/3}$, $T(010) = z^{1/3}$, $T(011) = T(110) = T(101) = z^{2/3}$ and $T(111) = z$. The eigenvalue of T is the solution of the equation

$$\lambda^3 = z(\lambda^2 + \lambda + y). \tag{57}$$

The density of sites is determined by equation (14) which gives

$$\rho = \frac{\lambda^2 + \lambda + y}{\lambda^2 + 2\lambda + 3y}, \tag{58}$$

whereas the density ρ_{00} of pairs of vacancies is determined by $\rho_{00} = (y/\lambda)\partial\lambda/\partial y$ which gives

$$\rho_{00} = \frac{y}{\lambda^2 + 2\lambda + 3y}. \tag{59}$$

From these two equations we find λ and y as functions of ρ and ρ_{00}:

$$\lambda = \frac{2\rho + \rho_{00} - 1}{1 - \rho - 2\rho_{00}}, \tag{60}$$

$$y = \frac{\rho_{00}(2\rho + \rho_{00} - 1)}{(1 - \rho - 2\rho_{00})^2}. \tag{61}$$

Substituting these two results into (57), we get

$$z = \frac{(2\rho + \rho_{00} - 1)^2}{\rho(1 - \rho - 2\rho_{00})}. \tag{62}$$

In the active state the density of active sites is determined by $\rho_a = 2\rho_{1100} + 2\rho_{1101} + 2\rho_{10100} = 2\rho_{110} + 2\rho_{10100}$. The quantity ρ_{110} is determined by modifying the element $T(110)$ as done in the previous section. The result is

$$\rho_{110} = \frac{(1 - \rho - \rho_{00})(2\rho + \rho_{00} - 1)}{\rho}. \tag{63}$$
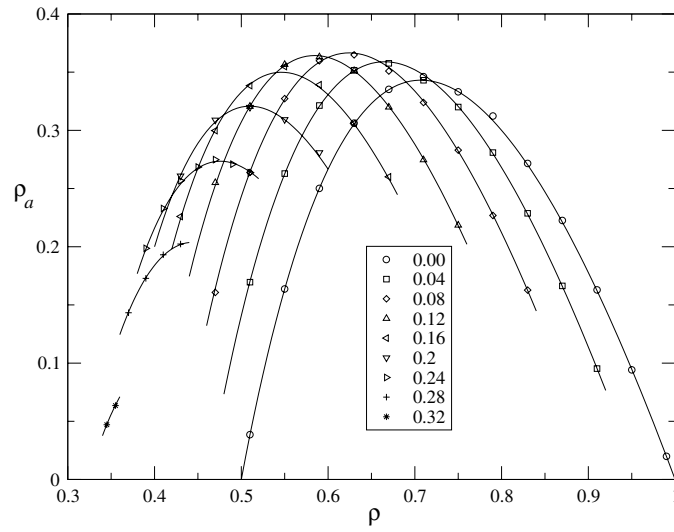
11

Figure 5. Density $\rho_a = 2\rho_{110} + 2\rho_{10100}$ of active particles as a function of the density ρ of particles for the model 4 as given by equations (63) and (64) (continuous line) and by simulation (symbols) for several values of ρ_{00}.

Again the calculation of ρ_{10101} needs the introduction of a pseudo-matrix involving five sites in a row. After straightforward but cumbersome algebraic steps we may find the result

$$\rho_{10100} = \frac{\rho_{00}(1 - \rho - 2\rho_{00})}{\rho}, \tag{64}$$

from which we get the desired expression ρ_a for the density of active sites in terms of the density of particles and density of pairs of vacancies. Both expressions (63) and (64) are valid in the active region of the phase diagram defined by $1 - 2\rho \leqslant \rho_{00} \leqslant (1 - \rho)/2$ and shown in figure 4. The density of active sites as a function of ρ for several values of ρ_{00} is shown in figure 5.

7. Conclusion

We have solved exactly four one-dimensional particle conservation lattice models with infinitely many absorbing states. This allowed us to determine the critical behavior of each model from which we obtained the exponent $\beta = 1$ for all models with the exception of one of them for which $\beta = 2$. The exact solution was obtained by the use of a transfer matrix technique and the transition occurred when the largest eigenvalue of the matrix vanished. The crucial point that allowed us to find the exact solution was the fact that inside the active subspace all configurations were equally probable making up a microcanonical ensemble. Although this result alone does not guarantee the finding of an exact solution in any dimension, it actually does in one dimension as we have shown here.

Appendix

Suppose that a partition function

$$Z = \sum_{\eta} F(\eta_1, \eta_2, \eta_3, \ldots, \eta_L), \tag{A.1}$$

where the summation over the variables $(\eta_1, \eta_2, \ldots, \eta_L) = \eta$, is such that $F(\eta)$ can be written as the product

$$F(\eta) = \prod_{i=1}^{L} T(\eta_i, \eta_{i+1}, \ldots, \eta_{i+k}), \tag{A.2}$$

where periodic boundary conditions have been used. The variables η_i take p values. In the models we have considered here $p = 2$. When $k = 1$ the quantities $T(\eta_1, \eta_2)$ may be interpreted as the elements of a $p \times p$ square matrix T. When $k > 1$ we can no longer assert this and for this reason we call T a pseudo-matrix and $T(\eta_1, \eta_2, \ldots, \eta_k, \eta_{k+1})$ its elements. However, it is always possible to set up a genuine $p^k \times p^k$ square matrix $\hat{T}$, associated with T, by defining its elements as

$$\hat{T}(\eta_1, \eta_2, \ldots, \eta_k; \eta_1', \eta_2', \ldots, \eta_k')$$
$$= T(\eta_1, \eta_2, \ldots, \eta_k, \eta_{k+1}) \, \delta(\eta_1', \eta_2) \ldots \delta(\eta_{k-1}', \eta_k) \delta(\eta_k', \eta_{k+1}), \tag{A.3}$$

where $\delta(\eta_1, \eta_2)$ is the Kronecker delta. From this definition this relation follows

$$T(\eta_1, \eta_2, \ldots, \eta_k, \eta_{k+1}) = \sum_{\eta_1'} \ldots \sum_{\eta_k'} \hat{T}(\eta_1, \eta_2, \ldots, \eta_k; \eta_1', \eta_2', \ldots, \eta_k'). \tag{A.4}$$

Using these results in equation (A.2) it is straightforward to show that

$$Z = \operatorname{Tr} \hat{T}^L. \tag{A.5}$$

Since

$$\operatorname{Tr} \hat{T}^L = \sum_r \lambda_r^L, \tag{A.6}$$

the calculation of the partition function is reduced to finding the eigenvalues λ_r of $\hat{T}$. In the text, when we say 'the eigenvalue of T' we mean 'the eigenvalue of the associated matrix $\hat{T}$'.

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